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An Application of the Newsvendor Model to Production Theory under Demand Uncertainty

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Abstract

Traditional production theory assumes that supply and demand are always balanced and all products are sold. However, in the present study, we investigate a model of production that allows for surplus and shortage in sales of output under uncertainty of quantity demanded, generalizing the model by introducing a salvage price for surplus and a penalty cost for shortage. This model is an application of the newsvendor model to production theory. According to traditional production theory, the level of output depends on prices and production technology. However, we explain that it also depends on the probability distribution of demand and the nature of the product with respect to maintaining its quality and reputation, as well as on the degree of damage due to the cost of loss of trust by supply shortages.

We conclude that output and profit are smaller under demand uncertainty than under certainty and compare output and profit across several cases under uncertainty. We classify the conditions for the existence of optimal output under certainty and uncertainty into three cases. We also show that adding demand uncertainty to the model provides the robustness necessary to ensure the existence of finite optimal output. In addition, we perform a numerical analysis and illustrate our discussion with a diagram showing the relationship between output under demand uncertainty and progress in production technology.

Our model under demand uncertainty, which assumes that firms seek to maximize expected profit and set only the quantity of output, represents a simple and special case of the more generalized model. However, this assumption leads to new results that will be helpful in facilitating analyses of production theory under demand uncertainty.

Keywords: Production Theory, Demand Uncertainty, Newsvendor Model, Inventory

Ⅰ　Introduction

Production theory and traditional economics studies usually assume that supply and demand are always balanced, with the result that there are no goods remaining unsold due to insufficient demand and no lost opportunities for sales due to supply shortages. In the real world, however, it is questionable whether markets are that simple or firms are that optimistic.

According to traditional production theory, the level of output depends on price and production technology. However, in the present study, by distinguishing between supply and demand in sales and considering excess demand and shortage of supply under demand uncertainty, we explain how the level of output also depends on the probability distribution of demand and the nature of the product with respect to maintaining its quality and reputation, as well as on damage due to the cost of loss of trust by supply shortages. We demonstrate that both output and profit are smaller under demand uncertainty than under certainty. With respect to profit, our result is in contrast to the result obtained by Oi (1961), who finds that optimal profit is larger under price uncertainty than under certainty.

This analysis, which distinguishes between supply and demand in sales, is treated as the newsvendor problem in inventory studies, such as Arrow, Harris, and Marschak (1951), Arrow, Karlin, and Scarf (1958), and Hadley and Whitin (1963) in early research, as well in numerous subsequent studies. In inventory studies, the newsvendor model has been applied to perishable or seasonal goods that decrease in value after the sales period. However, when considering discounts due to various types of deterioration or obsolescence, even non-perishable and non-seasonable goods will tend to decrease in value after the sales period. Moreover, when possible fines or the cost of loss of trust due to goods being out of stock are taken into account, the newsvendor model can be applied to an even wider range of goods. This study treats such goods. Eppen (1979), Chen and Lin (1989), Chang and Lin (1991), Cherikh (2000), and Lin, Chen, and Hsieh (2001) apply a salvage price for unsold goods and a penalty cost for out-of-stock goods to the inventory model in order to study the effects of centralizing inventory systems.

Practically, Mills (1959) and Hymans (1966) discuss production theory within a framework that distinguishes between supply and demand in sales. Mills (1959) assumes that firms set both the price and quantity of output to maximize expected profit and obtain results that compare optimal prices between certainty and uncertainty. Mills (1959) uses the difference in demand from the average as a stochastic variable, whereas we use quantity demanded itself as a stochastic variable to facilitate our analysis. Hymans (1966) demonstrates that output is less under uncertainty than under certainty, on the assumption that expected utility is maximized. These researchers made the application of a model that distinguishes between supply and demand in sales, namely the newsvendor model, to production theory. However, their arguments are complex and difficult to apply to other analyses. In contrast, we assume that firms set only the quantity of output and maximize the expected profit. Although in that sense our model represents a special case, it opens the way to simple discussions and yields new results that are easy to apply to other analyses under the same assumptions.

In constructing our model, we begin by applying the simple newsvendor model with only a sales price to production with the aim of maximizing expected profit, then we proceed to a generalized model by adding a salvage price for unsold goods and a penalty cost for out-of-stock goods. This is a novel approach. We also compare output and profit across the newsvendor models and obtain new conclusions. We classify the conditions for the existence of optimal output under certainty and uncertainty into three cases and show that the newsvendor model provides the robustness necessary for the model of production to ensure the existence of finite optimal output even when the cost function is concave, which is the opposite case to convex.

We perform a numerical analysis assuming a normal distribution for the quantity demanded and compare the optimal output and the maximum profit across the cases under demand certainty and uncertainty. This numerical analysis illustrates the above discussion graphically and shows that the higher the production technology, the larger the reducing effect of demand uncertainty on output, and that the output is less affected by technological progress under demand uncertainty than under certainty.

Ⅱ A simple model of production applying the newsvendor model

In this model, we assume that the quantity demanded always matches output under demand certainty, while it is stochastic under uncertainty. We let p be the product price of a good, q be the output, and $c(q)$ be the cost function. We assume that $c(q)$ is strictly increasing, convex, and twice differentiable, and $0 = c(0)$. Thus, $0 < c'(q)$ and $0 \leq c''(q)$ for any positive q unless otherwise specified. We let x be the non-negative stochastic quantity demanded for the good under uncertainty in a period and subject to a probability density function $f(x)$. We let q^m be the maximum value of q that satisfies $\int_0^q f(x)dx = 0$ and q^M be the minimum value of q that satisfies $\int_q^\infty f(x)dx = 0$.[1] Thus, $0 \leq q^m < q^M$ holds.[2] It is assumed that $0 < f(x)$ for $q^m < x < q^M$ and $c'(q^m) < p < c'(\infty)$.[3] The meaning of $c'(q^m) < p < c'(\infty)$ with respect to the existence of the solution for q under demand certainty and uncertainty will be discussed later.

First, we consider a simple model with only p, and later we generalize the model by introducing a

1) q^M can be infinite.
2) If $q^m = q^M$, x is no longer a random variable.
3) $p < c'(\infty)$ precisely means that some sufficiently large value of q satisfies $p < c'(q)$.

salvage price for surplus and a penalty cost for shortage. Under demand uncertainty, if the demand is smaller than the output, the quantity sold equals the demand, while if the demand is larger than the output, the quantity sold equals the output. We call this premise the newsvendor model and apply it to production theory. In the simple model applying the newsvendor model with only p, we let R be the expected profit and $R(q)$ be its function of q, which is given by

$$R = R(q) = p\left(\int_0^q xf(x)\,dx + \int_q^\infty qf(x)\,dx\right) - c(q). \tag{1}$$

The first-order condition to maximize $R(q)$ is given by

$$dR/dq = p\int_q^\infty f(x)\,dx - c'(q) = 0. \tag{2}$$

The second-order condition is satisfied as follows:

$$d^2R/dq^2 = -pf(x) - c''(q) \le 0. \tag{3}$$

Under demand certainty, the first-order condition for profit maximization is $p = c'(q)$. By comparing this equation with Eq. (2), we can intuitively grasp that the optimal output is larger under certainty than under uncertainty. Nevertheless, we rigorously verify the existence of the solution and compare the optimal outputs under demand certainty and uncertainty.

We let q^c and q^p be the optimal outputs under demand certainty and uncertainty, respectively. Thus, $p = c'(q^c)$ holds and q^p satisfies Eq. (2). We show that $q^m < q^p < q^c$ holds. Since $c'(q^m)/p < 1 < c'(\infty)/p$ from the assumption, it follows that q^c satisfying $1 = c'(q^c)/p$ exists and $q^m < q^c < \infty$ holds from the continuity of $c'(q)$. Letting $D(q) = dR/dq = p\int_q^\infty f(x)dx - c'(q) = p - c'(q) - p\int_0^q f(x)dx$, $0 < D(q^m) = p - c'(q^m)$ and $0 > D(q^c) = -p\int_0^{q^c} f(x)dx$ hold since $q^m < q^c$. Since $D(q^p) = 0$, it follows that q^p exists in the range of $q^m < q^p < q^c$. $D(q^M) = -c'(q^M) < 0$ also holds, so $q^m < q^p < q^M$ and $0 < \int_{q^p}^\infty f(x)dx < 1$ hold. It is indeterminate whether q^c or q^M is larger.

From the above, we obtain the following proposition:

Proposition 1

In the simple newsvendor model with only p, $q^c > q^p$ and $q^m < q^p < q^M$ hold.

Equation (2) shows how output is determined by the probability distribution of demand in addition to price and production technology, and that the firm should make $\int_q^\infty f(x)dx$, which can be called the sold-out rate, equal to $c'(q)/p$ at the optimal output.

Figure 1 illustrates the solutions of q under demand certainty and uncertainty. Taking q on the horizontal axis, and $c'(q)$ and $p\int_q^\infty f(x)dx$ on the vertical axis, the intersection point of the two curves of $c'(q)$ and $p\int_q^\infty f(x)dx$ marks the solution of (2), which is the optimal output under uncertainty, while the intersection point of the curve of $c'(q)$ and the horizontal line from p on the vertical axis is the optimal output under certainty. By comparing the two intersection points, Proposition 1 can be confirmed. As long as $c'(q^m) < p < c'(\infty)$ holds, no matter how large the mean of the distribution is or how far to the right the distribution spreads, $q^c > q^p$ always holds.

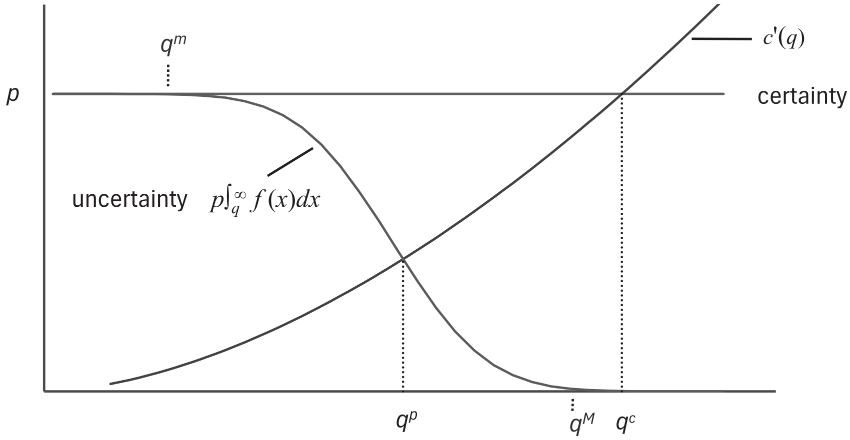

Figure 1. The optimal values of q under certainty (q^c) and uncertainty (q^p) in a simple model

The conditions for the existence of a solution for q under demand certainty and uncertainty are classified into the three cases, depending on the position of $c'(q)$ relative to the horizontal line from p on the vertical axis and q^m in Figure 1. The three cases and their implications are described below. Figure 1 corresponds to Case 3).

Case 1)　$p < c'(0)$

$c'(q)$ is completely above the horizontal line from p on the vertical axis. This implies that productivity is too low, so solutions do not exist under either certainty or uncertainty.

Case 2)　$p = c'(q)$ for some q in $0 \leq q \leq q^m$

$c'(q)$ intersects the line segment to the right of p on the vertical axis and to the left of q^m. A solution exists under certainty, but not under uncertainty.

Case 3)　$p > c'(q^m)$

$c'(q)$ intersects the line segment from p to the right of q^m. Solutions exist under both certainty and uncertainty, where the solution under certainty is larger than that under uncertainty.

In the present study, we examine when solutions exist under uncertainty, so we treat Case 3). Traditional theory treats Case 2).

Next, let us consider the case where $c(q)$ is concave (the opposite of the convex case), meaning that $c''(q) \leq 0$. Under certainty, without $\int_q^\infty f(x)dx$, $dR/dq > 0$ can hold at (2) as q approaches ∞, which implies that there is no finite optimal output. Conversely, under uncertainty, $dR(q)/dq < 0$ holds for any q in $q_0 < q$, where q_0 is a positive value, since $\int_q^\infty f(x)dx$ goes to 0 as q goes to q^M.[4] This implies that $R(q)$ reaches a maximum value for q somewhere in the range of $q \leq q_0$, and that uncertainty in the quantity demanded, i.e., the newsvendor model, provides the robustness necessary for optimal output to exist.

Ⅲ A generalized model with a salvage price and a penalty cost

To generalize our model, we introduce a salvage price for surplus and a penalty cost for shortage. We assume that when demand is less than supply, a certain amount of revenue is generated for the surplus, $q - x$, at a salvage price s per unit, and when demand is greater than supply, a certain penalty cost is generated for the shortage, $x - q$, at a penalty cost v per unit. Our model assumes that $0 \leq s < p$, $0 \leq v$, and that the salvage price and unit cost are constant. Letting R^G be the generalized expected profit, adding s and v to R of (1), and letting $R^G(q)$ be its function of q, we obtain the following equation:

$$R^G = R^G(q) = \int_0^q (px + s(q-x))f(x)\,dx + \int_q^\infty (pq - v(x-q))f(x)\,dx - c(q). \tag{4}$$

Let $A = s\int_0^q f(x)dx + (p+v)\int_q^\infty f(x)dx$ and $A(q)$ denote its function of q. Let $\alpha = p - s + v$, where $\alpha > 0$ holds. $dA/dq = (-p + s - v)f(q) = -\alpha f(q)$ holds where $dA/dq < 0$ for $q^m < q < q^M$ and $dA/dq = 0$ for $q \leq q^m$ or $q^M \leq q$. The first-order condition for the maximum $R^G(q)$ is given by

$$\begin{aligned} dR^G/dq &= s\int_0^q f(x)\,dx + (p+v)\int_q^\infty f(x)\,dx - c'(q) \\ &= A(q) - c'(q) = 0. \end{aligned} \tag{5}$$

The second-order condition is satisfied as follows:

$$d^2R^G/dq^2 = -\alpha f(x) - c''(q) \leq 0. \tag{6}$$

Equation (5) shows that, in addition to price and production technology, and the probability distribution of demand, the output also depends on the nature of the product with respect to maintaining its quality and reputation, as well as on the degree of damage caused by supply shortages, which are represented by s and v.

In addition to q^p, which is the optimal output value when s=0 and v=0, we assign that value to q^s when s>0 and v=0, to q^v when s=0 and v>0, and to q^{sv} when s>0 and v>0. We also let q^G represent all types of optimal q in the generalized model. Similarly, we let A^p be A when s=0 and v=0, A^s be A when s>0 and v=0, A^v be A when s=0 and v>0, and A^{sv} be A when v>0 and s>0. We let A represent the

4) Strictly speaking, for example, even when $c''(q) \leq 0$, if $c_0 \leq c'(q)$ holds as q approaches q^M, where c_0 is some small positive value, $dR(q)/dq < 0$ holds for any q in $q_0 < q$.

generalized types of A. We let $A^p(q)$, $A^s(q)$, $A^v(q)$, and $A^{sv}(q)$ denote the functions of A^p, A^s, A^v, and A^{sv} for q, as follows:

$$A^p = A^p(q) = p\int_q^\infty f(x)\,dx = p - p\int_0^q f(x)\,dx\,, \tag{7a}$$

$$A^s = A^s(q) = s\int_0^q f(x)\,dx + p\int_q^\infty f(x)\,dx = p - (p-s)\int_0^q f(x)\,dx\,, \tag{7b}$$

$$A^v = A^v(q) = (p+v)\int_q^\infty f(x)\,dx = p - p\int_0^q f(x)\,dx + v\int_q^\infty f(x)\,dx\,, \tag{7c}$$

$$A^{sv} = A^{sv}(q) = s\int_0^q f(x)\,dx + (p+v)\int_q^\infty f(x)\,dx = p - (p-s)\int_0^q f(x)\,dx + v\int_q^\infty f(x)\,dx. \tag{7d}$$

We define $R^p(q)$ to be $R^G(q)$ when s=0 and v=0, $R^s(q)$ to be $R^G(q)$ when s>0 and v=0, $R^v(q)$ to be $R^G(q)$ when s=0 and v>0, $R^{sv}(q)$ to be $R^G(q)$ when s>0 and v>0. We let R^p, R^s, R^v, and R^{sv} be the maximum values of $R^p(q)$, $R^s(q)$, $R^v(q)$, and $R^{sv}(q)$, respectively. Thus, $R^p = R^p(q^p)$, $R^s = R^s(q^s)$, $R^v = R^v(q^v)$, and $R^{sv} = R^{sv}(q^{sv})$.

From (5), it follows that $A^p(q^p) = c'(q^p)$, $A^s(q^s) = c'(q^s)$, $A^v(q^v) = c'(q^v)$, and $A^{sv}(q^{sv}) = c'(q^{sv})$. We verify the existence of q^G and show its range.

We let $D^G(q) = dR^G(q)/dq = A(q) - c'(q)$ and q^{Mc} represent the larger of q^M and q^c. Thus, $D^G(q^m) = p + v - c'(q^m) > 0$ and $D^G(q^{Mc}) = s - c'(q^{Mc}) < p - c'(q^{Mc}) \le p - c'(q^c) = 0$. Therefore, q^G exists from the continuity of $D^G(q)$, and $q^m < q^G < q^{Mc}$ and $0 < \int_0^{q^G} f(x)dx$ hold. We let $D^v(q) = dR^v(q)/dq = A^v(q) - c'(q)$. Thus, $D^v(q^m) = p + v - c'(q^m) > 0$ and $D^v(q^M) = -c'(q^M) < 0$. The range of q^v is $q^m < q^v < q^M$, which is narrower than or equal to that of q^G.

Next, we compare q^p, q^s, q^v, and q^{sv}. In making comparisons between pairs of these quantities, we assume that p, s, and v have the same value when the values of both are positive. Figure 2 in the later section provides an intuitive illustration of the results. However, in our analysis, we compare their values rigorously.

Since q^G exists in the range of $q^m < q^G < q^{Mc}$ and q^v exists in the range of $q^m < q^v < q^M$, using $0 \le c''(q)$ and $dA^G/dq \le 0$, we can say the following from (7a), (7b), (7c), and (7d).

For any q in $q^m < q \le q^p$, since $0 < \int_0^q f(x)dx$, $c'(q) \le c'(q^p) = A^p(q^p) \le A^p(q) < A^s(q)$ holds. Thus, $A^s(q) = c'(q)$ does not hold and q^s does not exist in the range of $q^m < q \le q^p$. Accordingly, $q^p < q^s$. Similarly, for any q in $q^m < q \le q^s$, $c'(q) \le c'(q^s) = A^s(q^s) \le A^s(q) < p$ holds. Thus, $p = c'(q)$ does not hold and q^c does not exist in the range of $q^m < q \le q^s$. Accordingly, $q^s < q^c$. For any q in $q^m < q \le q^v$, $c'(q) \le c'(q^v) = A^v(q^v) \le A^v(q) < A^{sv}(q)$ holds. Thus, $A^{sv}(q) = c'(q)$ does not hold and q^{sv} does not exist in the range of $q^m < q \le q^v$. Accordingly, $q^v < q^{sv}$. For any q in $q^m < q \le q^p$, $c'(q) \le c'(q^p) = A^p(q^p) \le A^p(q) <$

$A^v(q)$ holds, since $0<\int_q^\infty f(x)dx$ as $q^p < q^M$. Thus, $c'(q) = A^v(q)$ does not hold and q^v does not exist in the range of $q^m < q \le q^p$. Accordingly, $q^p < q^v$.

To compare q^s and q^{sv}, we consider the two cases of $q^c \le q^M$ and $q^c > q^M$. In the case of $q^c \le q^M$, for any q in $q^m < q \le q^s$, $c'(q) \le c'(q^s) = A^s(q^s) \le A^s(q) < A^{sv}(q)$ holds, since $q^s < q^M = q^{MC}$. Thus, $c'(q) = A^{sv}(q)$ does not hold and q^{sv} does not exist in the range of $q^m < q \le q^s$. Accordingly, $q^s < q^{sv}$ holds. Next, we consider the case of $q^c > q^M$. If $q^s < q^M$, for any q in $q^m < q \le q^s$, $c'(q) \le c'(q^s) = A^s(q^s) \le A^s(q) < A^{sv}(q)$ holds. Thus, $c'(q) = A^{sv}(q)$ does not hold and q^{sv} does not exist in the range of $q^m < q \le q^s$. Accordingly, $q^s < q^{sv}$. If $q^M \le q^s$, for any q in $q^m < q < q^M$, $c'(q) \le c'(q^s) = A^s(q^s) < A^s(q) < A^{sv}(q)$ holds. Thus, $c'(q) = A^{sv}(q)$ does not hold and q^{sv} does not exist in the range of $q^m < q < q^M$. Accordingly, $q^M \le q^{sv}$.

Therefore, we obtain the following proposition:

Proposition 2

In the generalized newsvendor model that includes s and v in addition to p, $q^m < q^G < q^{Mc}$ holds, in detail, $q^p < q^s < q^c$, $q^p < q^v < q^{sv}$, and $q^v < q^M$ hold.

Moreover, $q^s < q^{sv}$ holds in the case of $q^c \le q^M$, and $q^s < q^{sv}$ holds if $q^s < q^M$ while $q^M \le q^{sv}$ holds if $q^M \le q^s$ in the case of $q^c > q^M$.

$q^s < q^c$ means that adding any positive value of s to the simple model with only p still ensures that q is smaller under uncertainty than under certainty. Proposition 2 also implies that adding s or v to the simple model increases output. Adding s to the generalized model with only v increases output, and adding v to the generalized model with only s increases output in the cases indicated. In addition, if $q^M \le q^s$ in the case of $q^c > q^M$, $c'(q^s) = A^s(q^s) = s = A^{sv}(q^{sv}) = c'(q^{sv})$ holds. Thus, under the assumption of $0 \le c''(q)$, whether q^s and q^{sv} equal or are different is indeterminate. However, q^s equals q^{sv} under the assumption of $0 < c''(q)$. The above discussion and the results deriving from Proposition 2 will be illustrated later in Figure 2.

Ⅳ　Comparison of maximum profit under demand certainty and uncertainty

We compare the maximum profit under demand certainty and uncertainty at the same price and unit cost. $R^G(q^G)$ is the maximum value of R^G in the generalized model under uncertainty. We let R^c be the maximum profit under certainty. Transforming (4), we obtain the following equation:

$$R^G(q^G) = pq^G - c(q^G) - (p-s)\int_0^{q^G}(q^G - x)f(x)dx + v\int_{q^G}^{\infty}(q^G - x)f(x)dx \tag{8}$$
$$< pq^c - c(q^c) = R^c.$$

The above inequality holds since $0 \le q^G - x$ in the first integral and $q^G - x < 0$ in the second integral, and $pq^G - c(q^G) \leq pq^c - c(q^c)$ holds since q^c maximizes $pq - c(q)$.

Therefore, for the same price and unit cost, we obtain the following proposition.

Proposition 3

The maximum expected profit under demand uncertainty, $R^G(q^G)$, is less than the maximum profit under certainty, R^c.

This conclusion contrasts with the finding by Oi (1961) that profit is larger under price uncertainty than under certainty.

We compare the maximum values of each type of R^G. In making comparisons between pairs of these quantities, we assume that p, s, and v have the same value when the values of both are positive. By using the results of Proposition 1 and Proposition 2, we obtain the following from (4). Since $R^p(q^v) - v\int_{q^v}^{\infty}(x - q^v)f(x)dx = R^v(q^v)$ and $0 < \int_{q^v}^{\infty} f(x)dx$, $R^v = R^v(q^v) < R^p(q^v) \leq R^p(q^p) = R^p$ holds. Since $R^p(q^p) + s\int_0^{q^p}(q^p - x)f(x)dx = R^s(q^p)$ and $0 < \int_0^{q^p} f(x)dx$, $R^p = R^p(q^p) < R^s(q^p) \leq R^s(q^s) = R^s$ holds. Since $R^v(q^v) + s\int_0^{q^v}(q^v - x)f(x)dx = R^{sv}(q^v)$ and $0 < \int_0^{q^v} f(x)dx$, $R^v = R^v(q^v) < R^{sv}(q^v) \leq R^{sv}(q^{sv}) = R^{sv}$ holds. Since $R^s(q^{sv}) - v\int_{q^{sv}}^{\infty}(x - q^{sv})f(x)dx = R^{sv}(q^{sv})$ and $0 \leq \int_{q^{sv}}^{\infty} f(x)dx$, $R^{sv} = R^{sv}(q^{sv}) \leq R^s(q^{sv}) \leq R^s(q^s) = R^s$ holds. Therefore, it holds that $R^v < R^p < R^s$, $R^v < R^{sv} \leq R^s$.

From (4) and (5), we obtain the following equation:

$$R^G(q^G) = A\, q^G - c(q^G) + (p - s)\int_0^{q^G} xf(x)dx - v\int_{q^G}^{\infty} xf(x)dx \tag{9}$$

$$= c'(q^G)q^G - c(q^G) + (p - s)\int_0^{q^G} xf(x)dx - v\int_{q^G}^{\infty} xf(x)dx$$

$$\geq (p - s)\int_0^{q^G} xf(x)dx - v\int_{q^G}^{\infty} xf(x)dx$$

Since $c(0) = 0$, $c(q^G) = c(0) + q^G c'(\theta)$ holds by Taylor expansion, and $c'(\theta) \leq c'(q^G)$, for $0 < \theta < q^G$, $c(q^G) \leq q^G c'(q^G)$ holds. Thus, we obtain the inequality in (9). By adding the previous results and applying (9) to each type of R^G, we obtain the following proposition by:

Proposition 4

It holds that $R^v < R^p < R^s$, $R^v < R^{sv} \leq R^s$, $0 < p\int_0^{q^p} xf(x)dx \leq R^p$, $0 < (p - s)\int_0^{q^s} xf(x)dx \leq R^s$, $p\int_0^{q^v} xf(x)dx - v\int_{q^v}^{\infty} xf(x)dx \leq R^v$, and $(p - s)\int_0^{q^{sv}} xf(x)dx - v\int_{q^{sv}}^{\infty} xf(x)dx \leq R^{sv}$.

The inequalities in comparing each type of R^G imply that the addition of s to the model under uncertainty increases the maximum expected profit and that the addition of v either decreases or else does not increase the maximum expected profit depending on the case. The remaining inequalities represent the lower limit of the maximum expected profit expressed as a definite integral or 0.

Ⅴ　Numerical analysis

Assuming a normal distribution for $f(x)$ for the sake of computation, we numerically calculate the optimum output and maximum profit under demand certainty and uncertainty. We ignore the probability that x is less than 0 as this is very low. We assume that p = 100, and that s = 40 and v = 25, when they are positive. By replacing $A(q)$ with $A^p(q)$, $A^s(q)$, $A^v(q)$, and $A^{sv}(q)$, we solve (5) for each case.

Assuming one-input production, we let $c(q) = c_1 q^{\beta}$ be the cost function, where c_1 and β are positive coefficients.[5)]　Setting $1 \leq \beta$ makes the cost function convex. (5) is specified as follows:

$$A(q) - c'(q) = A(q) - c_1 \beta q^{\beta-1} = 0\,. \tag{10}$$

Taking the same variables at the axes as in Figure 1, Figure 2 illustrates the optimal values of q under certainty and uncertainty.

The intersection points of the curve of $c'(q)$ and the horizontal line from p on the vertical axis, and the curves of A^p, A^s, A^v, and A^{sv} represent the solutions to (10), namely q^c, q^p, q^s, q^v, and q^{sv}. We assume that $q^s > q^v$, $q^c \leq q^M$, and $q^s > q^M$ in Figure 2.

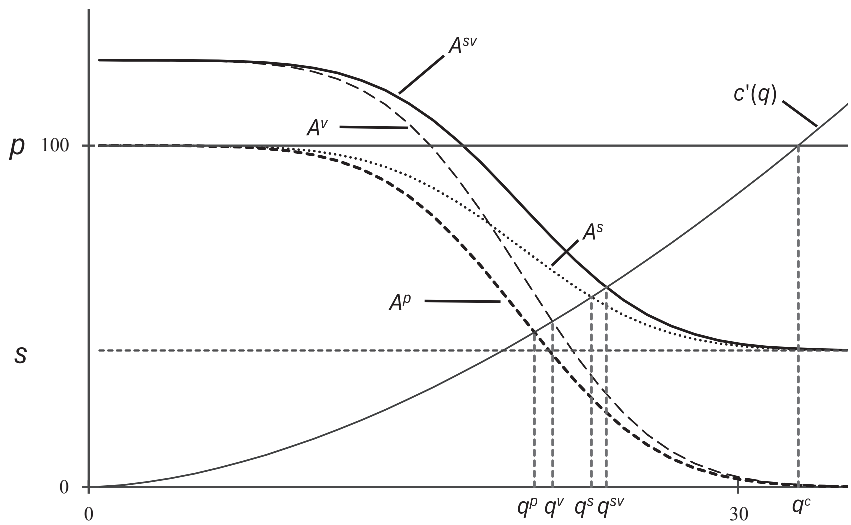

Figure 2. Optimal values of q under certainty and uncertainty calculated with N (μ = 20, σ = 5) for $f(x)$ and c_1 = 0.098, β = 2.7

5) By letting $q = a I^b$ be the production function and $c = c_2 I$, where c is the cost, I is the input, a, b, and c_2 are the positive coefficients with $b \leq 1$, $I = (q/a)^{1/b}$ holds and the cost function is described as $c(q) = c_2 (q/a)^{1/b} = c_2 a^{-1/b} q^{1/b}$. By letting $c_1 = c_2 a^{-1/b}$ and $\beta = 1/b$, we obtain $c(q) = c_1 q^{\beta}$ with $\beta \geq 1$.

Table 1 q^* and R^* calculated with a normal distribution for $f(x)$ (μ =20, σ =5)

[1] The case of c_1= 0.098 β=2.7 | [2] The case of c_1= 0.098 β=2.6 | rate of change

	q^*	*ratio* q^*	R^*	*ratio* R^*	q^*	*ratio* q^*	R^*	*ratio* R^*	q^*	R^*
q^c	32.82	1.000	2,066	1.000	41.80	1.000	2,572	1.000	0.274	0.245
q^p	20.59	0.627	1,483	0.718	21.86	0.523	1,582	0.615	0.062	0.066
q^s	23.23	0.708	1,604	0.776	26.02	0.623	1,755	0.682	0.120	0.094
q^v	21.43	0.653	1,445	0.699	22.63	0.541	1,555	0.605	0.056	0.076
q^{sv}	23.92	0.729	1,587	0.768	26.50	0.634	1,749	0.680	0.108	0.102

q^* is the optimal q and R^* is the maximum R under certain and uncertainty in each case.

ratio q^* is the ratio of q^* under uncertainty to q^c in each case.

ratio R^* is the ratio of R^* under uncertainty to R^c in each case.

rate of change is the rate of change of q^* or R^* when β changes from 2.7 to 2.6.

The calculated results are presented in Table 1. [1] in Table 1 shows the results for $c_1 = 0.098$ and $\beta = 2.7$, while [2] shows the results for $c_1 = 0.098$ and $\beta = 2.6$, where β represents the difference in productivity. The smaller the value of β, the higher the productivity and the lower the curve of $c'(q)$ in Figure 2. Figure 2 corresponds to [1] in Table 1.

We let $f(x)$ be a normal density function, $N(\mu,\sigma^2)$, of which $\mu = 20$ and $\sigma = 5$, and let $\phi(\varepsilon)$ be the standard normal density function, $\varepsilon = (x-\mu)/\sigma$, and $\eta = (q-\mu)/\sigma$, where $\varepsilon \sim \phi(\varepsilon)$. Under the above assumptions, by changing the variable of the integral from x to ε, we can transform (4) into (11). See the Appendix for the proof.

$$R^G(q) = (p-s)\mu + s\,q - (p-s+v)\left((\mu-q)\int_\eta^\infty \phi(\varepsilon)d\varepsilon + \sigma\phi(\eta)\right) - c(q). \tag{11}$$

In Table 1, q^* is the optimal value of q and R^* is the maximum profit in each case. Under uncertainty, q^* is obtained by solving (10), where $A(q) = s + \alpha\int_q^\infty f(x)dx = s + \alpha\int_\eta^\infty \phi(\varepsilon)d\varepsilon$ and $\alpha = p - s + v$.[6] R^* is obtained by substituting q^* into (11). Under certainty, q^c is obtained by solving $p = c'(q^c)$, and $R^c = p\,q^c - c(q^c)$, which is R^* under uncertainty, is obtained. The calculated results in Table 1 are consistent with these inequalities that are theoretically predicted by Proposition 2.[7]

In each case, *ratio* q^* is the ratio of q^* under uncertainty to q^c and *ratio* R^* is the ratio of R^* under uncertainty to R^c. *rate of change* of q^* is the rate of change of q^* when β changes from 2.7 to 2.6, which is obtained by subtracting q^* when $\beta = 2.7$ from q^* when $\beta = 2.6$ and dividing the difference by

6) q^* is calculated numerically by computer, not analytically, from (10).

7) In Figure 2, we assume a normal distribution for $f(x)$, so q^m and q^M are infinite and do not appear in Figure 2, and $q^c \le q^M$ holds.

q^* when $\beta = 2.7$, in each case. *rate of change* of R^* is calculated in the same way.

Examining *ratio* q^* and *ratio* R^* in Table 1 shows that the impacts of demand uncertainty on production are considerable. Comparing the results shown in Table 1 [1] and [2], we see that the smaller β is, the larger q^* and R^* are, but the smaller *ratio* q^* and *ratio* R^* are. This means that higher production technology increases production and profits, but also increases the impact of demand uncertainty on the decline in production.

From Figure 2 we can infer that, as β deceases, the movement of the intersection to the right is smaller under uncertainty than under certainty. This effect is reflected in *rate of change* of q^* and R^*, which are smaller under uncertainty than certainty. This implies that the impact of progress in production technology on output, which is indicated by a smaller value of β, is weaker under uncertainty than under certainty. These findings are from numerical examples, but they are also generally considered to be true.

Ⅵ　Related studies

Measuring demand uncertainty by price, many studies have discussed whether demand uncertainty decreases or increases output or investment under various assumptions. Oi (1961) demonstrates that a larger price uncertainty results in a larger expected profit and a higher expected utility if the firm has a constant or increasing marginal utility. Baron (1970) shows that optimal output is a non-increasing function of a firm's index of risk-aversion, and Sandmo (1971) demonstrates that output is smaller under price uncertainty than under certainty. Batra and Ullah (1974) indicates that higher price uncertainty leads to a smaller output if the firm is risk-averse in the two-input model, and Abel (1983) shows that higher price uncertainty leads to a higher current investment rate. Leland (1972), Holthausen (1976), and Hau (2004) assume that, if firms set either the price or the quantity of demand, the other becomes a random variable. However, unless the quantity of demand is treated as stochastic, it is not possible to distinguish between supply and demand in sales.

Research on demand uncertainty frequently assumes that firms maximize expected utility; for example, Hymans (1966), Baron (1970), Sandmo (1971), Leland (1972), Batra and Ullah (1974), Holthausen (1976), and Hau (2004). Conversely, Abel (1983), Hartman (1972), Oi (1961), and Mills (1962) assume that firms maximize expected profit. In the present study, we assume that firms maximize their expected profit, not their expected utility. Which is more useful and meaningful as a target for maximization under demand uncertainty in production theory: expected profit or expected utility? As Mills (1962) and Hymans (1966) argue, maximization of expected profit is a simple and special case of the maximization of expected utility. However, the following observation in Sandmo

(1971, pp. 65-66) should not be neglected: "We shall assume the firm's attitude toward risk can be summarized by a von Neumann-Morgenstern utility function. This may be a strong assumption, because in many firms, decisions are typically taken by a group of individuals, and group preferences may not always satisfy the transitive axiom required for the existence of a utility function." Additionally, the results obtained using expected utility depend on the attitude toward risk, and although markets are composed of three types of firms: risk averse, risk neutral, and risk seeking, the assumption of maximizing expected profit is independent of a firm's attitude to risk and is very common among firms in markets.

Ⅶ Summary and concluding remarks

This study is an application of the newsvendor model to production theory. Traditionally, the output level has been considered to depend on price, cost, and production technology. However, our arguments have attempted to explain how it also depends on the probability distribution of demand and the nature of the product with respect to maintaining its quality and reputation, as well as on the degree of damage due to the cost of loss of trust by supply shortages.

We reached the basic conclusion that optimal output and maximum profit are smaller under demand uncertainty than under certainty and showed that it is true, no matter how large the mean of the distribution is or how far to the right the distribution is spread. We classified the conditions for the existence of optimal output under demand certainty and uncertainty into three cases. Furthermore, we showed that adding demand uncertainty to the model of production provides the robustness necessary to ensure the existence of finite optimal output even when the cost function is concave.

We conducted a numerical analysis using a normal distribution for demand. This numerical analysis calculates and compares the optimal output and the maximum profit between demand certainty and uncertainty, and illustrates the above discussion graphically, while showing that the higher the production technology, the larger the effect of uncertainty on output reduction. However, output is less affected by technological progress under demand uncertainty than under certainty. These findings are from numerical examples, but they are also generally considered to be true.

We have assumed that firms maximize expected profit and set only the quantity of output, which can be understood as a simple and special case of maximizing expected utility and setting both the price and quantity of output. However, this assumption simplifies the discussion and the results, which is helpful in facilitating analyses under these assumptions.

Appendix

Assuming a normal distribution for $f(x)$, we show that (4) is rewritten as (11). First, we prove that $\int_q^\infty xf(x)dx = \mu\int_\eta^\infty \phi(\varepsilon)d\varepsilon + \sigma\int_\eta^\infty \varepsilon\phi(\varepsilon)d\varepsilon$.

$\phi(\varepsilon) = f(x)dx/d\varepsilon = f(\mu + \varepsilon\sigma)\sigma$ holds, since $x = \mu + \sigma\varepsilon$, $x \sim f(x)$, and $\varepsilon \sim \phi(\varepsilon)$. By changing the variable of the integral from x to ε, we obtain the following equation for any value of a and b ($a < b$):

$$\int_a^b xf(x)dx = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} xf(x)(dx/d\varepsilon)d\varepsilon \tag{A1}$$

$$= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} (\mu + \varepsilon\sigma)f(\mu + \varepsilon\sigma)\sigma d\varepsilon$$

$$= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} (\mu + \varepsilon\sigma)\phi(\varepsilon)d\varepsilon$$

$$= \mu\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \phi(\varepsilon)d\varepsilon + \sigma\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \varepsilon\phi(\varepsilon)d\varepsilon\,.$$

By letting $a = q$ and $b = \infty$, from $\eta = (q - \mu)/\sigma$, we obtain the following equation:

$$\int_q^\infty xf(x)dx = \mu\int_\eta^\infty \phi(\varepsilon)d\varepsilon + \sigma\int_\eta^\infty \varepsilon\phi(\varepsilon)d\varepsilon\,. \tag{A2}$$

Thus, since $\int_q^\infty f(x)dx = \int_\eta^\infty \phi(\varepsilon)d\varepsilon$, (4) can be rewritten as follows:

$$R^G(q) = (p - s)\mu + sq + (p - s + v)\left(q\int_q^\infty f(x)\,dx - \int_q^\infty xf(x)\,dx\right) - c(q) \tag{A3}$$

$$= (p - s)\mu + s\,q - (p - s + v)\left((\mu - q)\int_\eta^\infty \phi(\varepsilon)d\varepsilon + \sigma\int_\eta^\infty \varepsilon\phi(\varepsilon)d\varepsilon\right) - c(q).$$

Since $\int_\eta^\infty \varepsilon\phi(\varepsilon)d\varepsilon = \phi(\eta)$ holds from the property of the standard normal distribution, (4) can be rewritten as follows (this is (11)):

$$R^G(q) = (p - s)\mu + s\,q - (p - s + v)\left((\mu - q)\int_\eta^\infty \phi(\varepsilon)d\varepsilon + \sigma\,\phi(\eta)\right) - c(q). \tag{A4}$$

JEL Classification: D21 D24 D81 D84

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